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The equations of the geometrically non-linear theory of elasticity and momentless shells for arbitrary displacements☆

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ABSTRACT

To validate earlier results for the case of arbitrary deformations and displacements in orthogonal curvilinear coordinates, kinematic and static relations of the non-linear theory of elasticity are set up which, in the limit of small deformations, lead, unlike the known relations, to correct and consistent relations. The same relations are also constructed for momentless shells of general form for the case of arbitrary displacements and deformations on the basis of which the problem of the static instability of a cylindrical shell with closed ends, made of a linearly elastic material and under conditions of an internal pressure (the problem of the inflation of a cylinder), is considered. It is shown that, in the case of momentless shells, the components of the true sheat stresses are symmetrical, unlike the three-dimensional case. All the above-mentioned relations are constructed for the loading of deformable bodies both by conservative external forces of constant directions and, also, by two types of "following" forces.

It has been established[1,2] that the relations of the geometrically non-linear theory of elasticity for small deformations and arbitrary displacements constructed by Novozhilov, which are used in all the scientific and educational literature on the mechanics of deformable solids as being absolutely correct and rigorously substantiated, are incorrect. On account of this, a non-contradictory version of the kinematic relations has been constructed in the quadratic approximation which, unlike the known relations, are correct, consistent and do not lead to the appearance of "false" bifurcations in the solution of specific geometrically non-linear problems. Based on these relations, consistent versions of the geometrically non-linear theory of momentless shells[3] of thin shells of general form with moments and of straight rods[4] have also been constructed for the case of small deformations and arbitrary displacements which enable one, within the limits of the corresponding linearized equations, to reveal a number of new non-classical forms of the loss of stability of cylindrical shells[3–5] and rods[3] in the case of some forms or other of their subcritical stress-strain state.

With the aim of validating the results in Refs 1 and 2 below, in addition to the results obtained by Novozhilov (see Ref.,[6] for example), kinematic and static relations of the non-linear theory of elasticity are set up for the case of arbitrary deformations and displacements which, in the limit of small deformations lead to correct and consistent relations.

The same relations are also constructed for momentless shells of general form for arbitrary displacements and deformations.

1. Equations of the theory of elasticity for arbitrary displacements

Finite deformations. We will assume that a deformable rigid body, which has a volume V prior to deformation, is referred to an orthogonal curvilinear system of coordinates x^α (α = 1,2,3) and is bounded by the surfaces $x^\alpha = x^\alpha_-$, $x^\alpha = x^\alpha_+$, which will henceforth be denoted by $\Omega^\pm_\alpha$. Here and everywhere below, the indices α, β and γ take the values of 1, 2, 3 and summation from 1 to 3 is carried out over the repeated index γ. We define the position of a certain point of the body $M \in V$ with coordinates x^α by specifying the radius vector $\mathbf{R}(x^\alpha)$ which, as a result of the deformation of the body, gains an increment due to the displacement vector $\mathbf{u} = u_\alpha \mathbf{I}_\alpha$, where $\mathbf{I}_\alpha = \mathbf{R}_\alpha/|\mathbf{R}_\alpha| = \mathbf{R}_\alpha/\mathbf{H}_\alpha$ are unit vectors satisfying the equalities $\mathbf{I}_\alpha \mathbf{I}_\beta = \delta_{\alpha\beta}$ ($\delta_{\alpha\beta}$ is the Kronecker delta and $\mathbf{R}_\alpha = \partial \mathbf{R}/\partial x^\alpha$) and for which the differentiation formulae

$$\mathbf{I}_{1,1} = -H_{1,2}\mathbf{I}_2/H_2 - H_{1,3}\mathbf{I}_3/H_3, \quad \mathbf{I}_{1,2} = H_{2,1}\mathbf{I}_2/H_1; \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.1}$$

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hold. After the body has been deformed, the expressions

$$\mathbf{R}_\alpha^* = H_\alpha(\delta_{\alpha\gamma} + e_{\alpha\gamma})\mathbf{l}_\gamma \tag{1.2}$$

$$e_{11} = H_1^{-1}(u_{1,1} + H_2^{-1}H_{1,2}u_2 + H_3^{-1}H_{1,3}u_3), \quad e_{12} = H_1^{-1}(u_{2,1} - H_2^{-1}H_{1,2}u_1); \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.3}$$

are obtained, using formula (1.1), for the principal basis vectors $\mathbf{R}_\alpha^* = \partial(\mathbf{R} + \mathbf{u})/\partial x^\alpha$, which enable as to determine the components of the principal metric tensor using the formulae

$$g_{\alpha\beta}^* = \mathbf{R}_\alpha^*\mathbf{R}_\beta^* = H_\alpha H_\beta(\delta_{\alpha\beta} + 2\varepsilon_{\alpha\beta})$$

According to Novozhilov,[6] the quantities

$$2\varepsilon_{\alpha\beta} = e_{\alpha\beta} + e_{\beta\alpha} + e_{\alpha\gamma}e_{\beta\gamma} \tag{1.4}$$

are called the stress components of the deformations. In present-day deformable body mechanics, the quantities[7,8]

$$\eta_{\alpha\beta} = (g_{\alpha\beta}^* - g_{\alpha\beta})/2 = H_\alpha H_\beta \varepsilon_{\alpha\beta}, \quad \text{here} \quad g_{\alpha\beta} = \mathbf{R}_\alpha\mathbf{R}_\beta$$

are also taken as a measure of both finite and small deformations in the case of arbitrary displacements.

However, the quantities

$$\varepsilon_\alpha = (dl_\alpha^* - dl_\alpha)/dl_\alpha = \sqrt{1 + 2\varepsilon_{\alpha\alpha}} - 1 \tag{1.5}$$

where $dl_\alpha = H_\alpha dx^\alpha$, $dl_\alpha^* = H_\alpha(1 + 2\varepsilon_{\alpha\alpha})^{1/2}dx^\alpha = H_\alpha^* dx^\alpha$ are elements of the lengths of arcs in the coordinate lines x^α, x_α^* before and after the deformation of the body, are the true[6] tensile deformations.

When account is taken of formula (1.5) for the unit vectors $\mathbf{l}_\alpha^*$, directed along the tangents to the coordinate lines x_*^α, the relations

$$\begin{aligned}\mathbf{l}_\alpha^* &= \mathbf{R}_\alpha^*/\left|\mathbf{R}_\alpha^*\right| = \mathbf{R}_\alpha^*/H_\alpha^* = (1 + 2\varepsilon_{\alpha\alpha})^{-1/2}(\delta_{\alpha\gamma} + 2e_{\alpha\gamma})\mathbf{l}_\gamma = \\ &= (1 + \varepsilon_\alpha)^{-1}(\delta_{\alpha\gamma} + e_{\alpha\gamma})\mathbf{l}_\gamma\end{aligned} \tag{1.6}$$

are obtained and, by using these to determine of the shear deformations $\sin\gamma_{\alpha\beta}$, the kinematic relations

$$\begin{aligned}\sin\gamma_{\alpha\beta} &= \cos(\pi/2 - \gamma_{\alpha\beta}) = \mathbf{l}_\alpha^*\mathbf{l}_\beta^* = 2\varepsilon_{\alpha\beta}(1 + 2\varepsilon_{\alpha\alpha})^{-1/2}(1 + 2\varepsilon_{\beta\beta})^{-1/2} = \\ &= 2\varepsilon_{\alpha\beta}(1 + \varepsilon_\alpha)^{-1}(1 + \varepsilon_\beta)^{-1}, \quad \alpha \neq \beta\end{aligned} \tag{1.7}$$

are established.

Prior to its deformation, we separate out a curvilinear parallelepiped from the body with orthogonal sides dl_1, dl_2, dl_3, and faces x^α = const with areas

$$S_1 = H_2H_3dx^2dx^3, \quad \overrightarrow{\underleftarrow{1, 2, 3}}$$

and a volume equal to $dV = H_1H_2H_3dx_1dx_2dx_3$. After deformation of the body, the above-mentioned faces will have areas s_1^*, s_2^*, s_3^*, which are connected to the areas S_1, S_2 and S_3 by the relations[6]

$$S_1^*/S_1 = \sqrt{(1 + 2\varepsilon_{22})(1 + 2\varepsilon_{33}) - (2\varepsilon_{23})^2}, \quad \overrightarrow{\underleftarrow{1, 2, 3}}$$

$$\text{a} \quad dV_* = \sqrt{g_*}dx^1dx^2dx^3, \ \text{где} \ g_* = \det(g_{\alpha\beta}^*). \tag{1.8}$$

In the areas S_α^*, we introduce into the treatment the vectors of the true[6] stresses $\boldsymbol{\sigma}_\alpha = \sigma_{\alpha\gamma}\mathbf{l}_\gamma$, divided by the units of the areas S_α^* and the generalized[6] stress vectors σ_α^*, divided by the units of the areas S_α. They must be related by the equalities[6]

$$\boldsymbol{\sigma}_\alpha^* = \frac{S_\alpha^*}{S_\alpha}\sigma_{\alpha\gamma}\mathbf{l}_\gamma^* \tag{1.9}$$

Inserting expressions (1.6) here instead of $\mathbf{l}_\gamma^*$ and introducing the notation

$$\sigma_{\alpha\beta}^* = \frac{S_\alpha^*}{S_\alpha}\frac{\sigma_{\alpha\beta}}{1 + \varepsilon_\beta}, \quad s_{\alpha\beta}^* = \sigma_{\alpha\gamma}^*(\delta_{\gamma\beta} + e_{\gamma\beta}) \tag{1.10}$$

we obtain

$$\boldsymbol{\sigma}_\alpha^* = s_{\alpha\gamma}^*\mathbf{l}_\gamma \tag{1.11}$$

where, by definition, $s_{\alpha\beta}^*$ are the components of the stress vectors $\boldsymbol{\sigma}_\alpha^*$ divided by the units of the areas S_α in the projections onto the undeformed axes.

The relations

$$1 + 2\varepsilon_{\alpha\alpha} = (1+\varepsilon_\alpha)^2, \quad 2\varepsilon_{\alpha\beta} = (1+\varepsilon_\alpha)(1+\varepsilon_\beta)\sin\gamma_{\alpha\beta}, \quad \alpha \neq \beta \tag{1.12}$$

follow from relations (1.5) and (1.7) and, using these, formulae (1.8) can be represented in the form

$$S_1^*/S_1 = \Delta_{23}, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.13}$$

The notation

$$\Delta_{23} = (1+\varepsilon_2)(1+\varepsilon_3)\cos\gamma_{23}$$

has been adopted here.

Since, by definition,[6] $\sigma^*_{\alpha\beta} = \sigma^*_{\beta\alpha}$, then, starting from relations (1.10) and taking account of formulae (1.13), we arrive at the formulae

$$\sigma^*_{11} = \frac{\sigma_{11}\Delta_{23}}{1+\varepsilon_1}, \quad \sigma^*_{12} = \sigma_{12}(1+\varepsilon_3)\cos\gamma_{23} = \sigma_{21}(1+\varepsilon_3)\cos\gamma_{13}; \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.14}$$

which are very convenient for analysing the basic relations of the theory of elasticity in the case of small deformations.

The vector equilibrium equation

$$f^* = (H_2H_3\boldsymbol{\sigma}_1^*)_{,1} + (H_1H_3\boldsymbol{\sigma}_2^*)_{,2} + (H_1H_2\boldsymbol{\sigma}_3^*)_{,3} + H_1H_2H_3\mathbf{F}^* = 0 \tag{1.15}$$

must be satisfied in the curvilinear system of coordinates in the case of the stress vectors $\boldsymbol{\sigma}^*_\alpha$ and the vector of the bulk forces $F^* = F^*_\gamma \mathbf{l}_\gamma$, divided by the unit of volume dV, prior to deformation of the body, which were introduced into the treatment. The scalar equilibrium equations in the projections on the undeformed axes

$$f^*_\alpha + H_1H_2H_3F^*_\alpha = 0 \tag{1.16}$$

where

$$\begin{aligned} f^*_1 &= (H_2H_3s^*_{11})_{,1} + (H_1H_3s^*_{21})_{,2} + (H_1H_2s^*_{31})_{,3} + \\ &+ H_3H_{1,2}s^*_{12} + H_2H_{1,3}s^*_{13} - H_3H_{2,1}s^*_{22} - H_2H_{3,1}s^*_{33}, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \end{aligned} \tag{1.17}$$

follow from (1.15) when representations (1.11) and differentiation formulae (1.1) are used.

If the body has a canonical form and surface forces $\mathbf{p}^*_\alpha = p^*_{\alpha\gamma}\mathbf{l}_\gamma$, divided by the units of areas S_α, are specified at the points of its boundary surfaces $x^\alpha = x^\alpha_+, x^\alpha = x^\alpha_-$, then, in a static equilibrium state, the variational equation of the principle of possible displacements (the summation is from 1 to 3 over the index δ)

$$\iiint\limits_V (\sigma^*_{\alpha\gamma}\delta\varepsilon_{\delta\gamma} - F^*_\delta\delta u_\delta)dV - \iint\limits_{\Omega_\delta} \mathbf{p}^*_\delta \delta\mathbf{u} S_\delta \Big|_{x^\delta = x^\delta_-}^{x^\delta = x^\delta_+} = 0 \tag{1.18}$$

must be satisfied and, using relations (1.4), this equation becomes

$$(s^*_{\delta\gamma} - p^*_{\delta\gamma})\delta u_\gamma \Big|_{x^\delta = x^\delta_-}^{x^\delta = x^\delta_+} - \iiint\limits_V (f^*_\gamma + H_1H_2H_3F^*_\gamma)\delta u_\gamma dx^1dx^2dx^3 = 0 \tag{1.19}$$

The equilibrium equations (1.16) follow from this and, at the points of the boundary surfaces $x_\pm = x_\pm{}^\alpha$, static boundary condition of the form

$$s^*_{\alpha\beta} = p^*_{\alpha\beta} \text{ when } \delta u_\beta \neq 0 \tag{1.20}$$

if the forces $p^*_{\alpha\beta}$ are specified for constant directions of the vectors $\mathbf{p}^*_\alpha$ ("dead" forces[9]).

If representations of the form $\mathbf{p}^*_\alpha = q^*_{\alpha\gamma}\mathbf{l}^*_\gamma$ are adopted for the vectors $\mathbf{p}^*_\alpha$, in which the components $q^*_{\alpha\beta}$ are specified, then these surface forces will be assumed to be "following" forces of the first type. Using relations (1.6) between $p^*_{\alpha\beta}$, occurring in the boundary conditions (1.20) and between the components $q^*_{\alpha\beta}$, the relations

$$p^*_{\alpha\beta} = \xi_\gamma(\delta_{\gamma\beta} + e_{\gamma\beta})q^*_{\alpha\gamma}, \quad \xi_\gamma = (1+\varepsilon_\gamma)^{-1} \tag{1.21}$$

are established. Treatment of the case when "following" surface forces $\mathbf{q}^*_\alpha$ of the second type, given by the expansion

$$\mathbf{p}^*_1 = q^*_1\mathbf{n}^*_1 + t^*_{12}\mathbf{l}^*_2 + t^*_{13}\mathbf{l}^*_3, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.22}$$

where $\mathbf{n}^*_1$ is the unit vector normal to the undeformed face $x^1 = \text{const}$, act on the body is also of practical interest. Since the vectors $\mathbf{l}^*_2$ and $\mathbf{l}^*_3$ lie in the tangential plane to this face, then, by using relations (1.6), it is possible to obtain

$$\mathbf{n}^*_1 = \frac{\mathbf{l}^*_2 \times \mathbf{l}^*_3}{\sin(\mathbf{l}^*_2\mathbf{l}^*_3)} = \frac{\mathbf{l}^*_2 \times \mathbf{l}^*_3}{\cos\gamma_{23}} = \frac{l_{11}\mathbf{l}_1 + l_{12}\mathbf{l}_2 + l_{13}\mathbf{l}_3}{\Delta_{23}}, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.23}$$

where

$$l_{11} = (1+e_{22})(1+e_{33}) - e_{23}e_{32}, \quad l_{12} = e_{23}e_{31} - e_{21}(1+e_{33}),$$
$$l_{13} = e_{21}e_{32} - e_{31}(1+e_{22}); \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.24}$$

By substituting the expansion $\mathbf{p}_1^* = p_{1\gamma}^*\mathbf{I}_\gamma$ and relations (1.6) and (1.23) into representation (1.22), it is possible to establish the equalities

$$p_{11}^* = \frac{q_1^* l_{11}}{\Delta_{23}} + \frac{t_{12}^* e_{21}}{1+\varepsilon_2} + \frac{t_{13}^* e_{31}}{1+\varepsilon_3}, \quad p_{12}^* = \frac{q_1^* l_{12}}{\Delta_{23}} + \frac{t_{12}^*(1+e_{22})}{1+\varepsilon_2} + \frac{t_{13}^* e_{32}}{1+\varepsilon_3},$$
$$p_{13}^* = \frac{q_1^* l_{13}}{\Delta_{23}} + \frac{t_{12}^* e_{23}}{1+\varepsilon_2} + \frac{t_{13}^*(1+e_{33})}{1+\varepsilon_3}; \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.25}$$

which serve to formulate the static boundary conditions (1.20) in the case of the action of "following" surface loads of the second type.

For the subsequent analysis and simplification of the relations presented above, the treatment of the particular form of deformation of a body, at all points of which the equalities $e_{\alpha\beta} \equiv 0$ ($\alpha \neq \beta$) are satisfied when $F_\alpha^* = 0$ and, at the same time, $\gamma_{12} = \gamma_{13} = \gamma_{23} = 0$, is of fundamental importance. By virtue of these equalities, relations (1.4) take the form $2\varepsilon_{\alpha\beta} = 2e_{\alpha\alpha} = e_{\alpha\alpha}^2$.

Consequently, according to equalities (1.5), the formulae

$$\varepsilon_\alpha = e_{\alpha\alpha} \tag{1.26}$$

are exact in the case of arbitrary deformations and, at the same time,

$$s_{\alpha\beta}^* = 0, \quad \alpha \neq \beta, \quad s_{11}^* = (1+e_{22})(1+e_{33})\sigma_{11}, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.27}$$

by virtue of which the equilibrium equations (1.16) take the form

$$(H_2^* H_3^* \sigma_{11})_{,1} - H_3 H_{2,1}(1+e_{11})(1+e_{33})\sigma_{22} - H_2 H_{3,1}(1+e_{11})(1+e_{22})\sigma_{33} = 0, \ \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.28}$$

where $H_\alpha^* = (1+\varepsilon_\alpha)H_\alpha = (1+e_{\alpha\alpha})H_\alpha$ are the Lamé parameters in the undeformed state of the body.

Small deformations. Relations (1.5) cab be represented in the form $\varepsilon_\alpha(2+\varepsilon_\alpha) = 2\varepsilon_{\alpha\alpha}$, where the approximate kinematic relations

$$\varepsilon_\alpha \approx \varepsilon_{\alpha\alpha} = e_{\alpha\alpha} + (e_{\alpha1}^2 + e_{\alpha2}^2 + e_{\alpha3}^2)/2 \tag{1.29}$$

with an accuracy $2+\varepsilon_\alpha \approx 2$, follow from this in the case of small deformations of the extensions $\varepsilon_\alpha \ll 1$ and, with an accuracy $1+\varepsilon_\alpha \approx 1$, instead of (1.7) we will have the approximate relations

$$\sin\gamma_{1,2} \approx 2\varepsilon_{12} = (1+e_{11})e_{21} + (1+e_{22})e_{12} + e_{13}e_{23}, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.30}$$

which hold for arbitrary shear deformations. The simplified formulae

$$S_1^*/S_1 \approx \cos\gamma_{23}, \quad \sigma_{11}^* = \sigma_{11}\cos\gamma_{23}; \quad \overrightarrow{\underleftarrow{1, 2, 3}}$$

which follow from relations (1.13) and (1.14) with an accuracy $1+\varepsilon_\alpha \approx 1$, correspond to such a deformed state.

If, together with the assumptions that $\varepsilon_\alpha \ll 1$, it is also assumed that the shear angles are small and we put

$$\sin\gamma_{\alpha\beta} \approx \gamma_{\alpha\beta} = 2\varepsilon_{\alpha\beta}, \quad \cos\gamma_{\alpha\beta} \approx 1; \quad \alpha \neq \beta$$

we arrive at the approximate equalities

$$H_\alpha^* \approx H_\alpha, \quad dV_* \approx dV, \quad S_\alpha^*/S_\alpha \approx 1, \quad \sigma_{\alpha\beta}^* \approx \sigma_{\alpha\beta}, \quad s_{\alpha\beta}^* \approx s_{\alpha\beta} = \sigma_{\alpha\gamma}(\delta_{\gamma\beta} + e_{\gamma\beta}) \tag{1.31}$$

which, together with relations (1.29) and (1.30), are well known in the literature and are used as being absolutely correct in the case of small deformations when, in accordance with the approximate equalities (1.31), there is no practical sense in introducing differences between the components of the generalized stresses ($\sigma_{\alpha\beta}^*$) and the true ($\sigma_{\alpha\beta}$) stresses. However, when they are used in the case when $e_{\alpha\beta} = 0$, $\gamma_{\alpha\beta} = 0$ ($\alpha \neq \beta$), the equations

$$[H_2 H_3(1+e_{11})\sigma_{11}]_{,1} - H_3 H_{2,1}(1+e_{22})\sigma_{22} - H_2 H_{3,1}(1+e_{33})\sigma_{33} = 0, \ \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.32}$$

follow from Eq. (1.16) when account is taken of relations (1.17) while the equations

$$(H_2 H_3 \sigma_{11})_{,1} - H_3 H_{2,1}\sigma_{22} - H_2 H_{3,1}\sigma_{33} = 0, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.33}$$

which do not agree with Eq. (1.32), follow in the case of small deformations from Eq. (1.28), which are exact in the case of finite deformations for the case of a deformed state being considered.

The reason for this disagreement is simple: if, in the exact formulation of the problem, transformations of the form

$$\sigma_{11}^*\delta\varepsilon_{11} = \frac{S_1^*}{S_1}\frac{\sigma_{11}\delta\varepsilon_{11}}{1+\varepsilon_1} = \frac{S_1^*}{S_1}\frac{\sigma_{11}(1+e_{11})\delta e_{11}}{1+e_{11}} = \frac{S_1^*}{S_1}\sigma_{11}\delta e_{11} =$$
$$= (1+\varepsilon_2)(1+\varepsilon_3)\sigma_{11}\delta e_{11} = \sigma_{11}^*\delta e_{11} \quad (1.34)$$

hold for the terms in Eq. (1.18) in the limit of a deformed state, which is characterized by the equalities $e_{\alpha\beta} = 0$ ($\alpha \neq \beta$), then, within the limits of the approximation

$$1+\varepsilon_1 \approx 1, \quad S_1^*/S_1 \approx 1, \quad \varepsilon_1 \approx \varepsilon_{11} = e_{11} + e_{11}^2/2, \quad \delta\varepsilon_{11} = (1+e_{11})\delta e_{11}$$

the transformations

$$\frac{S_1^*}{S_1}\frac{\sigma_{11}\delta\varepsilon_{11}}{1+\varepsilon_1} \approx \sigma_{11}\delta\varepsilon_{11} = \sigma_{11}(1+e_{11})\delta e_{11} \quad (1.35)$$

hold and the use of these also leads to Eq. (1.32).

In order to remove the defect in the approximate relations (1.29) which has been established and in the relations $s_{\alpha\beta}^* = \sigma_{\alpha\gamma}(\delta_{\gamma\beta} + e_{\gamma\beta})$ which are obtained when they are used, it is sufficient to discard terms of the form $e_{\alpha\alpha}^2/2$, in relations (1.29), replacing them with the approximate relations[1,2]

$$\varepsilon_1 \approx \varepsilon_{11} = e_{11} + (e_{12}^2 + e_{13}^2)/2, \quad \overleftarrow{\underleftarrow{1, 2, 3}} \quad (1.36)$$

The relations for $s_{\alpha\beta}^*(\alpha \neq \beta)$ then take the form

$$s_{11}^* = \sigma_{11}^*(1+e_{11}) + \sigma_{12}^*e_{21} + \sigma_{13}^*e_{31} =$$
$$= \left[\sigma_{11}(1+e_{11})\frac{(1+\varepsilon_2)(1+\varepsilon_3)}{1+\varepsilon_1} + \sigma_{12}e_{21}(1+\varepsilon_3) + \sigma_{13}e_{31}(1+\varepsilon_2)\right]\cos\gamma_{23} \approx$$
$$\approx s_{11} \approx \sigma_{11} + \sigma_{12}e_{21} + \sigma_{13}e_{31}, \quad \overleftarrow{\underleftarrow{1, 2, 3}} \quad (1.37)$$

while the relations for $s_{\alpha\beta}$ ($\alpha \neq \beta$) remain unchanged. In the case of small deformations, similar simplifications also have to be carried out in relations (1.21) by representing them in the form

$$p_{11}^* \approx q_{11}^* + q_{12}^*e_{21} + q_{13}^*e_{31} \approx p_{11} \approx q_{11} + q_{12}e_{21} + q_{13}e_{31}, \quad \overrightarrow{\underleftarrow{1, 2, 3}}$$
$$p_{\alpha\beta} \approx (\delta_{\gamma\beta} + e_{\gamma\beta})q_{\alpha\gamma}, \quad \alpha \neq \beta \quad (1.38)$$

if following surface forces $\mathbf{p}_\alpha^* \approx \mathbf{p}_\alpha$ of the first kind act on the body. If, however, following forces of the second kind act, then it is necessary to represent relations (1.25), which are required in order to formulate the static boundary conditions, in the form

$$p_{11}^* \approx p_{11} \approx q_1 + t_{12}e_{21} + t_{13}e_{31}, \quad \overrightarrow{\underleftarrow{1, 2, 3}}; \quad p_{12}^* \approx q_1 l_{12} + t_{12}(1+e_{22}) + t_{13}e_{32} \quad (1.39)$$

by putting

$$l_{1\alpha}/\Delta_{23} \approx 1, \quad \overrightarrow{\underleftarrow{1, 2, 3}}$$

The relations

$$\mathbf{l}_1^* \approx \mathbf{l}_1 + e_{12}\mathbf{l}_2 + e_{13}\mathbf{l}_3, \quad \mathbf{n}_1 = \mathbf{l}_1 + l_{12}\mathbf{l}_2 + l_{13}\mathbf{l}_3; \quad \overrightarrow{\underleftarrow{1, 2, 3}} \quad (1.40)$$

following from relations (1.6) and (1.23), which, when $e_{\alpha\beta} = 0(\alpha \neq \beta)$, like relations (1.6) and (1.23), lead to the exact equalities $\mathbf{I}_\alpha^* = \mathbf{I}_\alpha$, $\mathbf{n}_\alpha = \mathbf{I}_\alpha$, are also consistent in the case of small deformations.

2. Geometrically non-linear equations of the theory of momentless shells in the case of arbitrary deformations

We refer the space of the shell, which has a thickness t, to the parametrization

$$\mathbf{R}(x^1, x^2, z) = \mathbf{r}(x^1, x^2) + z\mathbf{m}, \quad -t/2 \le z \le t/2$$

where $\mathbf{r} = \mathbf{r}(x^1, x^2)$ is the parametric equation of the middle surface σ, referred to the lines of principal curvatures, and $\mathbf{m}$ is the unit vector of the normal to σ which, with the unit vectors $\mathbf{I}_i = \mathbf{r}_i/|\mathbf{r}_i| = \mathbf{r}_i/A_i$, constitutes a right-handed trihedron at each point of the surface σ.

In the system of coordinates which has been adopted, the differentiation formulae

$$\mathbf{l}_{1,1} = -A_2^{-1}A_{1,2}\mathbf{l}_2 - A_1k_1\mathbf{m}, \quad \mathbf{l}_{1,2} = -A_1^{-1}A_{2,1}\mathbf{l}_2, \quad \mathbf{m}_{,1} = A_1k_1\mathbf{l}_1; \quad \overrightarrow{\underleftarrow{1,2}} \tag{2.1}$$

hold for the vectors $\mathbf{l}_i$, and m, where k_i are the principal curvatures of the surface σ, the parametric equation of which, after deformation of the shell, is expressed by the equality

$$\mathbf{r}^* = \mathbf{r} + \mathbf{u}(x^1, x^2) = \mathbf{r} + u_k\mathbf{l}_l + w\mathbf{m} \tag{2.2}$$

Henceforth, summation from 1 to 2 is carried out over a repeated index k or s.

Differentiating equality (2.2) with respect to x^i, using formulae (2.1), at each point M^* of the deformed surface σ_*, by analogy with the first part of Section 1, we successively find the vectors of the principal basis

$$\mathbf{r}_i^* = \partial\mathbf{r}^*/\partial x^i = A_i[(\delta_{ik} + e_{ik})\mathbf{l}_l + \omega_i\mathbf{m}] \tag{2.3}$$

the unit vectors

$$\mathbf{l}_i^* = (1+\varepsilon_i)^{-1}[(\delta_{ik} + e_{ik})\mathbf{l}_k + \omega_i\mathbf{m}] \tag{2.4}$$

the tensile deformations ε_i and the shear deformation $\sin\gamma_{12}$

$$\varepsilon_i = \sqrt{1+2\varepsilon_{ii}} - 1, \quad \sin\gamma_{12} = 2\varepsilon_{12}(1+\varepsilon_1)^{-1}(1+\varepsilon_2)^{-1} \tag{2.5}$$

where

$$2\varepsilon_{ik} = e_{ik} + e_{ki} + e_{is}e_{ks} + \omega_i\omega_k \tag{2.6}$$

$$e_{11} = A_1^{-1}(u_{1,1} + A_2^{-1}A_{1,2}u_2) + k_1w, \quad e_{12} = A_1^{-1}(u_{2,1} - A_2^{-1}A_{1,2}u_1),$$

$$\omega_1 = A_1^{-1}w_{,1} - k_1u_1; \quad \overrightarrow{\underleftarrow{1,2}} \tag{2.7}$$

By definition, the area of an element of the curvilinear quadrangle on the deformed surface σ_* is equal to $d\sigma_* = \sqrt{a_*}dx^1dx^2$, where $a_* = a_{11}^*a_{22}^* - a_{12}^*$ is the determinant of the metric tensor on σ_* with components

$$a_{ik}^* = \mathbf{r}_i^*\mathbf{r}_k^* = A_iA_k(\delta_{ik} + 2\varepsilon_{ik})$$

Consequently,

$$a_* = A_1^2A_2^2[(1+2\varepsilon_{11})(1+2\varepsilon_{22}) - 4\varepsilon_{12}^2] \tag{2.8}$$

However, according to equalities (2.5),

$$1 + 2\varepsilon_{ii} = (1+\varepsilon_i)^2, 4\varepsilon_{12}^2 = (1+\varepsilon_1)^2(1+\varepsilon_2)^2\sin^2\gamma_{12}$$

Hence,

$$d\sigma_* = A_1A_2\Delta_{12}dx^1dx^2; \quad \Delta_{12} = (1+\varepsilon_1)(1+\varepsilon_2)\cos\gamma_{12} \tag{2.9}$$

In normal sections of the deformed shell, corresponding to x^1 = const and x^2 = const, we have $\sigma_{13} = 0$, $\sigma_{23} = 0$, for the transverse components of the shear stresses in the case of momentless shells and $\sigma_{33} = 0$ within the limits of the assumption concerning the plane stress state. Consequently, $\gamma_{13} = \gamma_{23} = 0$, by virtue of which relations (1.4) take the form

$$\sigma_{11}^* = \sigma_{11}\frac{(1+\varepsilon_2)(1+\varepsilon_3)}{1+\varepsilon_1}, \quad \sigma_{22}^* = \sigma_{22}\frac{(1+\varepsilon_1)(1+\varepsilon_3)}{1+\varepsilon_2}$$

$$\sigma_{12}^* = \sigma_{12}(1+\varepsilon_3) = \sigma_{21}(1+\varepsilon_3) \tag{2.10}$$

It follows from this that not only the components of the generalized stresses but, also, the components of the true stresses are symmetric in the momentless stress state of a shell, that is, $\sigma_{12} = \sigma_{21}$.

We now introduce the vectors of the generalized linear forces $\mathbf{T}_1^*$ and $\mathbf{T}_2^*$, divided by the units of the lengths of the coordinate lines x^1 and x^2 of the undeformed middle surface σ. By definition,

$$\mathbf{T}_i^* = \int_{-t/2}^{t/2} \boldsymbol{\sigma}_i^* dz = \int_{-t/2}^{t/2} \sigma_{ik}^*\mathbf{l}_k^* dz = T_{ik}^*\mathbf{l}_k^*, \quad T_{ij}^* = \int_{-t/2}^{t/2} \sigma_{ij}^* dz = t\sigma_{ij}^* \tag{2.11}$$

In a similar way, at the points of the deformed surface σ_*, we introduce the vector of the external surface forces $\mathbf{X}^*$, divided by a unit of the area of the undeformed surface σ, which is associated with the analogous vector $\mathbf{X}$, divided by a unit of the area of the deformed surface σ_*, by the equality $\mathbf{X}^*d\sigma = \mathbf{X}d\sigma^*$. Since $d\sigma = A_1A_2dx^1dx^2$, and $d\sigma_*$ is defined by to formula (2.9), then

$$\mathbf{X}^* = \mathbf{X}(1+\varepsilon_1)(1+\varepsilon_2)\cos\gamma_{12} \tag{2.12}$$

Using relations (2.10), the expressions for T^*_{ij} can be represented in the form

$$T^*_{11} = T_{11}\frac{1+\varepsilon_2}{1+\varepsilon_1}, \quad T^*_{22} = T_{22}\frac{1+\varepsilon_1}{1+\varepsilon_2}, \quad T^*_{12} = T_{12} = T_{21} = T^*_{21} \tag{2.13}$$

where the forces

$$T_{ij} = t(1+\varepsilon_3)\sigma_{ij} = t^*\sigma_{ij}, \quad t^* = t(1+\varepsilon_3)$$

which are the components of the vectors $\mathbf{T}_i = T_{ik}\mathbf{l}^*_k$, have been introduced into the treatment.

By definition, $\mathbf{T}_1$ and $\mathbf{T}_2$ are vectors, divided by the units of the lengths of the coordinate lines x^1_*, x^2_* of the deformed surface σ_*. The relations

$$\mathbf{T}^*_1 = (1+\varepsilon_2)\mathbf{T}_1, \quad \mathbf{T}^*_2 = (1+\varepsilon_1)\mathbf{T}_2 \tag{2.14}$$

must therefore hold between them and the vectors $\mathbf{T}^*_i$.

Substituting their representations

$$\mathbf{T}_i = T_{ik}\mathbf{l}^*_k = \xi_k T_{ik}[(\delta_{ks}+e_{ks})\mathbf{l}_s + \omega_k\mathbf{m}], \quad \xi_k = (1+\varepsilon_k)^{-1} \tag{2.15}$$

here, instead of $\mathbf{T}_i$, we obtain the representations

$$\mathbf{T}^*_i = S^*_{ik}\mathbf{l}_k + S^*_{i3}\mathbf{m} \tag{2.16}$$

for $\mathbf{T}^*_i$ when formulae (2.13) are taken into account. The components of the forces S^*_{ij} and S^*_{i3} occurring here, which are defined by the formulae

$$S^*_{ij} = T^*_{is}(\delta_{sj}+e_{sj}), \quad S^*_{i3} = T^*_{is}\omega_s \tag{2.17}$$

are the projections of the vectors of the forces $\mathbf{T}^*_1$ and $\mathbf{T}^*_2$, divided by the units of the lengths of the coordinate lines x^2 and x^1, on σ in the directions of the unit vectors $\mathbf{l}_i$ and $\mathbf{m}$ on σ.

The vectors of the internal forces $\mathbf{T}^*_i$ and $\mathbf{T}^*_2$ and the surface loads $\mathbf{X}^*$ must satisfy the vector equilibrium equation

$$\mathbf{f}^* = \frac{\partial(A_2\mathbf{T}^*_1)}{dx^1} + \frac{\partial(A_1\mathbf{T}^*_2)}{dx^2} + A_1A_2\mathbf{X}^* = 0 \tag{2.18}$$

in the case of static deformation of the shell and the vector equation of motion (τ is the time)

$$\mathbf{f}^* = \rho t A_1 A_2 \frac{\partial^2\mathbf{u}}{\partial\tau^2} \tag{2.19}$$

in the case of a dynamic deformation process.

If it is assumed that a shell on σ is bounded by the contour lines $x^i = x^i_-$, $x^i = x^i_+$ and the contour forces $\mathbf{P}^*_1$ and $\mathbf{P}^*_2$, divided by the units of the lengths of the lines $x^1 = \text{const}$ and $x^2 = \text{const}$, are specified on them, then the internal forces $\mathbf{T}^*_1$ occurring in Eq. (2.18) must satisfy the static boundary conditions

$$\mathbf{T}^*_i = \mathbf{P}^*_i \text{ when } x^i = x^i_-, \quad x^i = x^*_+ \tag{2.20}$$

Together with the vectors $\mathbf{P}^*_i$, the vectors of the given forces $\mathbf{P}_i$, divided by the units of the lengths of the deformed contour lines on σ_* can also be introduced into the treatment. They are connected by the relations

$$P^*_1 = (1+\varepsilon_2)P_1, \quad P^*_2 = (1+\varepsilon_1)P_2 \tag{2.21}$$

which are analogous to relations (2.14). Then, by representing the vectors $\mathbf{P}_i$ in the form of the expansions

$$\mathbf{P}_i = Q_{ik}\mathbf{l}^*_k = \xi_k Q_{ik}[(\delta_{ks}+e_{ks})\mathbf{l}_s + \omega_k\mathbf{m}], \quad \xi_k = (1+\varepsilon_k)^{-1}$$

after introducing the external generalized contour forces Q^*_{ij} into the treatment using the formulae

$$Q^*_{11} = Q_{11}\frac{1+\varepsilon_2}{1+\varepsilon_3}, \quad Q^*_{22} = Q_{22}\frac{1+\varepsilon_1}{1+\varepsilon_2}, \quad Q^*_{12} = Q_{12}, \quad Q^*_{21} = Q_{21} \tag{2.22}$$

we arrive at the representations

$$\mathbf{P}^*_i = P^*_{is}\mathbf{l}_s + P^*_{i3}\mathbf{m}; \quad P^*_{ik} = Q^*_{is}(\delta_{sk}+e_{sk}), \quad P^*_{i3} = Q^*_{is}\omega_s \tag{2.23}$$

Here, the projections of the vectors $\mathbf{P}^*_i$ on the undeformed axes, divided by the units of the lengths of the corresponding contour lines on the undeformed middle surface σ, have been introduced into the treatment.

If the vector of the surface forces $\mathbf{X}^*$ is represented by the expansion

$$\mathbf{X}^* = \mathbf{X}^*_k\mathbf{l}_k + \mathbf{X}^*_3\mathbf{m} \tag{2.24}$$

then, when expressions (2.16) and (2.24) are substituted into Eq. (2.18) and the differentiation formulae (2.11) are used, we arrive at the static equilibrium equations

$$f_1^* = \frac{\partial(A_2S_{11}^*)}{\partial x^1} + \frac{\partial(A_1S_{21}^*)}{\partial x^2} - \frac{\partial A_2}{\partial x^1}S_{22}^* + \frac{\partial A_1}{\partial x^2}S_{12}^* + A_1A_2(k_1S_{13}^* - X_1^*) = 0, \quad \overrightarrow{\underleftarrow{1, 2}}$$

$$f_3^* = \frac{\partial(A_2S_{13}^*)}{\partial x^1} + \frac{\partial(A_1S_{23}^*)}{\partial x^2} - A_1A_2(k_1S_{11}^* + k_2S_{22}^* - X_3^*) = 0 \tag{2.25}$$

in projections on the undeformed axes. In these equations, the external surface forces are assumed to be specified, if the vector $\mathbf{X}^*$ does not change its direction during the deformation process (a "dead" surface load). The vector $\mathbf{X}^*$ is assumed to be a "following" vector if its components are specified in the projections on the unit tangential vectors $\mathbf{l}_i^*$ and the normal $\mathbf{M}^*$:

$$\mathbf{X}^* = Y_i^*\mathbf{l}_i^* + Y_3^*\mathbf{m}^* \tag{2.26}$$

where

$$\mathbf{m}^* = \frac{\mathbf{l}_1^* \times \mathbf{l}_2^*}{\cos\gamma_{12}} = \frac{E_1\mathbf{l}_1 + E_2\mathbf{l}_2 + E_3\mathbf{m}}{\Delta_{12}} \tag{2.27}$$

$$\mathbf{E}_1 = e_{12}\omega_2 - \omega_1(1 + e_{22}), \quad \overrightarrow{\underleftarrow{1, 2}}; \quad \mathbf{E}_3 = (1 + e_{11})(1 + \mathbf{e}_{22}) - \mathbf{e}_{12}\mathbf{e}_{21} \tag{2.28}$$

According to representations (2.24) and (2.26), when relations (2.4) and (2.27) between the components X_i^*, X_3^* and Y_i^*, Y_3^* are used, the relations

$$X_1^* = \frac{1 + e_{11}}{1 + \varepsilon_1}Y_1^* + \frac{e_{21}}{1 + \varepsilon_2}Y_2^* + \frac{E_1}{\Delta_{12}}Y_3^*, \quad \overrightarrow{\underleftarrow{1, 2}}$$

$$X_3^* = \frac{\omega_1}{1 + \varepsilon_1}Y_1^* + \frac{\omega_2}{1 + \varepsilon_2}Y_2^* + \frac{E_3}{\Delta_{12}}Y_3^* \tag{2.29}$$

are established in which the components Y_1^*, Y_2^*, Y_3^* are assumed to be specified. In particular, in the case of the hydrostatic action of an external pressure p^* on a shell, which remains normal to the surface σ_* during the shell deformation, the external forces in Eq. (2.25) will have the form

$$X_\alpha^* = -\frac{E_\alpha}{\Delta_{12}}p^*, \quad \alpha = 1, 2, 3 \tag{2.30}$$

since, for such a load, $Y_1^* = Y_2^* = 0$.

The static boundary conditions for the equilibrium equations on the contour lines $x^i = x_-^i$, $x^i = x_+^i$ are formulated as follows:

$$S_{i1}^* = P_{i1}^* \text{ when } \delta u_1 \neq 0, \quad S_{i2}^* = P_{i2}^* \text{ when } \delta u_2 \neq 0, \quad S_{i3}^* = P_{i3}^* \text{ when } \delta w \neq 0 \tag{2.31}$$

The values of P_{ik}^* and P_{i3}^* are assumed to be specified for constant directions of the vectors $\mathbf{P}_i^*$ and, under the action of contour "following" forces of the first type, they are determined using formulae (2.23). If, however, the vectors $\mathbf{P}_i^*$ belong to the second type, that is, they are given by the representations (relation (2.21) are taken into account)

$$\mathbf{P}_1^* = (1 + \varepsilon_2)(Q_1\mathbf{n}_1^* + Q_{12}\mathbf{l}_1^*), \quad \overrightarrow{\underleftarrow{1, 2}} \tag{2.32}$$

in which the unit vectors

$$\mathbf{n}_1^* = \mathbf{l}_2^* \times \mathbf{m}^*, \quad \mathbf{n}_2^* = \mathbf{m}^* \times \mathbf{l}_1^* \tag{2.33}$$

occur, which are normal to the lines $x_*^1 = x_+^1, x_*^2 = x_+^2$ and lie in the tangential planes to the deformed surface σ_*, then it is necessary to consider the components Q_1, Q_2, Q_{12}, Q_{21}, divided by the units of the lengths of the arcs of the deformed contour lines, as being specified. In order to express the components P_{ik}^* and P_{i3}^* occurring in the boundary conditions (2.31) in terms of them, we obtain the expressions

$$\mathbf{n}_1^* = \frac{n_{11}\mathbf{l}_1 + n_{12}\mathbf{l}_2 + n_{13}\mathbf{m}}{\tilde{\Delta}_{12}}, \quad \overrightarrow{\underleftarrow{1, 2}}, \quad \tilde{\Delta}_{12} = (1 + \varepsilon_2)\Delta_{12}, \tag{2.34}$$

for the vectors $\mathbf{n}_i^*$, where

$$n_{11} = E_3(1 + e_{22}) - \omega_2E_2, \quad n_{12} = E_1\omega_2 - e_{21}E_3, \quad n_{13} = e_{21}E_2 - E_1(1 + e_{22}); \overrightarrow{\underleftarrow{1, 2}} \tag{2.35}$$

Then, on substituting expressions (2.4) and (2.34) into expression (2.32), we arrive at the relations

$$P_{11}^* = \frac{n_{11}}{\Delta_{12}}Q_1 + e_{21}Q_{12}, \quad P_{12}^* = \frac{n_{12}}{\Delta_{12}}Q_1 + (1 + e_{22})Q_{12}, \quad P_{13}^* = \frac{n_{13}}{\Delta_{12}}Q_1 + \omega_2Q_{12}; \overrightarrow{\underleftarrow{1, 2}} \tag{2.36}$$

which are used to formulate boundary conditions (2.31) when "following" loads of the second type act on the contour lines of the shell.

As the simplest example of the application of the relations of general form, which have been derived above, we consider the problem of the stress-strain state of a circular cylindrical shell with closed ends acted upon by an external pressure. Since the stress-strain state of the shell is axil symmetric for such loading, then, within the limits of the momentless theory when account is taken of the obvious equalities $A_1 = 1$, $A_2 = R$, we arrive at the absolutely accurate kinematic relations

$$\varepsilon_1 = e_{11} = \frac{du_1}{dx}, \quad \varepsilon_2 = w_{22} = \frac{w}{R} \tag{2.37}$$

where u_1 and ε_1 and the displacement and deformation along the axial coordinate x, w is the bending, R is the radius of the middle surface and ε_2 is the relative extension in the circumferential direction of the shell. By virtue of the equalities

$$e_{12} = e_{21} = \omega_1 = \omega_2 = 0$$

formulae (2.28) lead to the equalities

$$E_1 = E_2 = 0, \quad E_3 = (1+e_{11})(1+e_{22})$$

and, by virtue of the equality $\gamma_{12} = 0$, when they are used taking account of relations (2.37), formulae (2.29) lead to the equalities

$$X_1^* = X_2^* = 0, \quad X_3^* = Y_3^* = p^*$$

where p^* is the internal pressure per unit area of σ prior to the deformation of the shell. If the pressure, per unit area of σ_* in the deformation process is denoted by p, then, by formulae (2.30), for X_3^*, we arrive at the equality ($X_3 = Y_3 = p$)

$$X_3^* = (1+e_{11})(1+e_{22})p \tag{2.38}$$

In the case considered, the equilibrium equations (2.25) take the form $dS_{11}^*/dx = 0$, $S_{22}^* = RX_3^*$ by virtue of the equalities $k_1 = 0$, $k_2 = R$. Consequently, $S_{11}^* = P_{11}^*$. But, by virtue of formulae (2.17), $S_{ii}^* = T_{ii}^*(1+e_{ii})$ and, by virtue of formulae (2.23), $P_{11}^* = Q_{11}^*(1+e_{11})$. Hence

$$T_{11}^* = Q_{11}^*, \quad T_{22}^* = \frac{RX_3^*}{1+e_{22}} \tag{2.39}$$

When account is taken of relations (2.13) and (2.22), the first equality of (2.39) is transformed into the equality $T_{11} = Q_{11}$. By definition,

$$Q_{11} = \frac{p\Omega_*}{2\pi R_*} = \frac{pR}{2}(1+e_{22})$$

Consequently,

$$T_{11} = \frac{pR}{2}(1+e_{22}) = \frac{pR_*}{2} \tag{2.40}$$

Here, R_* is the radius of the deformed middle surface of the shell.

Similarly, using relations (2.38) and (2.13), the formula

$$T_{22} = pR(1+e_{22}) = pR_* \tag{2.41}$$

follows from the second equality of (2.39).

If it is assumed that Hooke's law holds between the deformations and the true normal stresses, then, in the case of a plane stress state when account is taken of relations (2.37), we have the relations

$$\sigma_{11} = \tilde{E}_1(e_{11} + \upsilon_{21}e_{22}), \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}} \tag{2.42}$$

and when these are used

$$\varepsilon_3 = -\frac{\upsilon_{31}}{E_1}\sigma_{11} - \frac{\upsilon_{32}}{E_2}\sigma_{22} = -\tilde{\upsilon}_1 e_{11} - \tilde{\upsilon}_2 e_{22} \tag{2.43}$$

where E_i are the moduli of elasticity of the first series, υ_{ij} are Poisson's ratios and

$$\tilde{\upsilon}_1 = \frac{\upsilon_{31} + \upsilon_{32}\upsilon_{12}}{1 - \upsilon_{12}\upsilon_{21}}, \quad \tilde{E}_1 = \frac{E_1}{(1 - \upsilon_{12}\upsilon_{21})}; \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}}$$

However, in the case of finite deformations $\sigma_{ii} = T_{ii}/[t(1+\varepsilon_3)]$. Consequently,

$$T_{11} = t(1+\varepsilon_3)\sigma_{11} = t(1+\varepsilon_3)\tilde{E}_1(e_{11} + \upsilon_{21}e_{22}), \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}} \tag{2.44}$$

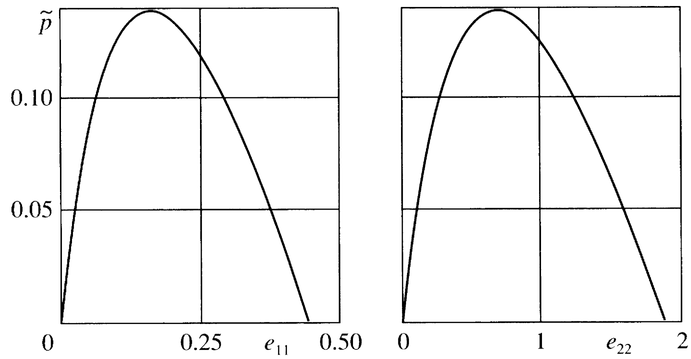

Fig. 1.

Substituting formulae (2.40) and (2.41) and relations (2.43) from (2.44), we strain the system of linear algebraic equations

$$pR(1+e_{22})/2 = t\tilde{E}_1(e_{11}+\upsilon_{21}e_{22})(1-\tilde{\upsilon}_1 e_{11}-\tilde{\upsilon}_2 e_{22})$$
$$pR(1+e_{22}) = t\tilde{E}_2(e_{22}+\upsilon_{12}e_{11})(1-\tilde{\upsilon}_1 e_{11}-\tilde{\upsilon}_2 e_{22}) \tag{2.45}$$

Dividing both sides of the second equation of (2.45) by two and subtracting it from the first equation, we obtain the relation

$$e_{11} = ge_{22}; \quad g = \frac{e(1-2\upsilon_{12})}{2-\upsilon_{12}e}, \quad e = \frac{\tilde{E}_2}{\tilde{E}_1} \tag{2.46}$$

and, using inequality (2.46) and introducing the dimensionless loading parameter $\tilde{p}$, the first equation is reduced to the quadratic equation

$$ae_{22}^2-(g+\upsilon_{21}-\tilde{p})e_{22}+\tilde{p} = 0 \tag{2.47}$$

where

$$a = g(\tilde{\upsilon}_2+\upsilon_{21}\tilde{\upsilon}_1)+\tilde{\upsilon}_2\upsilon_{21}+\tilde{\upsilon}_1 g^2, \quad \tilde{p} = \frac{pR}{2t\tilde{E}_1}$$

The relation

$$\tilde{p} = \frac{(g+\upsilon_{21})e_{22}-ae_{22}^2}{1+e_{22}} \tag{2.48}$$

follows from Eq. (2.47). Eq. (2.48) is shown for an isotropic shell on the right in Fig. 1 and the dependence of

$$\upsilon_{12} = \upsilon_{21} = \upsilon = 0.3, \quad \tilde{\upsilon}_1 = \tilde{\upsilon}_2 = \upsilon/(1-\upsilon), \quad e = 1, \quad g = (1-2\upsilon)/(2-\upsilon)$$

on $e_{11} = ge_{22}$ is shown on the left-hand side of the same figure. In Fig. 2 we show $\tilde{p}$ as a function of σ_{11}, σ_{22}, ε_3, where

$$\tilde{\sigma}_{11} = \sigma_{11}/\tilde{E}_1 = e_{11}+\upsilon_{21}e_{22}, \quad \tilde{\sigma}_{22} = e(e_{22}+\upsilon_{12}e_{11}), \quad \varepsilon_3 = -\tilde{\upsilon}_1 e_{11}-\tilde{\upsilon}_2 e_{22}$$

It can be seen that the relation $\tilde{p} = \tilde{p}(e_{22})$ as well as the other relations have points at which $\tilde{p}$ attains a maximum. According to relation (2.48) when $d\tilde{p}/de_{22} = 0$, for a positive maximum point we obtain

$$e_{22}^{*} = -1+\sqrt{1+g+\upsilon_{21}/a} \tag{2.49}$$

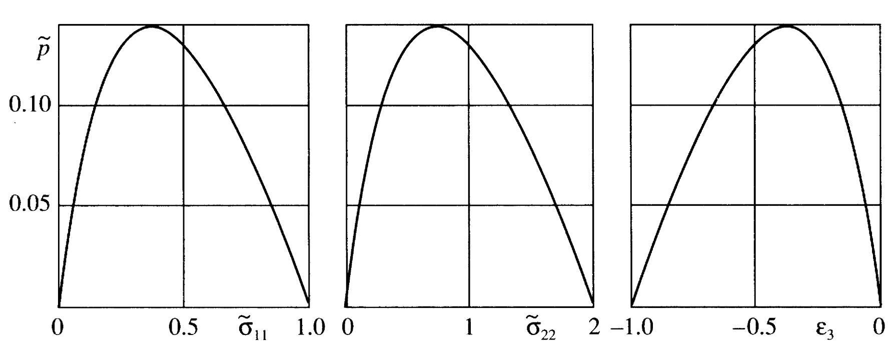

Fig. 2.

It should be noted that the problem considered above on the deformation of a cylinder under the action of an external pressure is completely analogous to problems by which the deformation of a rod under tension and the inflation of a sphere under the action of an internal pressure are described. The solutions found by Rzhanitsin and Feodos'ev are available with an analysis of the corresponding processes.[10] By analogy with them, the fixed value of the pressure $\tilde{p}_{\max}$ is the critical pressure, after the attainment of which the deformation of the shell become larger without any further increase in p. Up to the values e^*_{ii} and σ^*_{ii}, the deformation process is stable. If, however, the value of p is maintained at the level $p_{\max}$, the descending branches of the graphs shown in Figs. 1 and 2 are unstable.

If, however, gas is pumped into the shell up to a pressure $p_{\max}$, then, in the case of a subsequent unchanged amount of gas, the internal volume of the shell becomes larger in view of the increase in the deformations e_{22} and e_{11} on the descending branch of the process being investigated which, in its turn, leads to a reduction in the pressure within the shell.

3. Simplifications of the non-linear equations of the theory of momentless shells in the case of small deformations

In the case of small tensile deformations, when $\varepsilon_i \ll 1$, $1+\varepsilon_i \approx 1$, for shells which are in a momentless state, the approximate equalities

$$\sigma^*_{ij} \approx \sigma_{ij}, \quad i, j = 1, 2 \tag{3.1}$$

hold for any shear deformations $\sin\gamma_{12}$, since only the deformations ε_i appear in relation (2.10), unlike the three-dimensional case. In this case, the following formulae and equalities hold

$$dl^*_i \approx dl_i = A_i dx^i, \quad A^*_i \approx A_i, \quad T^*_{ij} \approx T_{ij}, \quad Q^*_{ij} \approx Q_{ij} \tag{3.2}$$

with an accuracy $1+\varepsilon_i \approx 1$.

However, the formulae (henceforth, as in Section 2, summation from 1 to 2 is carried out over a repeated index s or k) which are obtained from relations (2.17) and (2.23) with the same degree of accuracy

$$S^*_{ij} \approx S_{ij} = T_{is}(\delta_{sj} + e_{sj}), \quad P^*_{ij} \approx P_{ij} = Q_{is}(\delta_{sj} + e_{sj}) \tag{3.3}$$

only hold when $i \neq j$, that is, for S^*_{12}, S^*_{21}, P^*_{12}, P^*_{21} since, when

$$e_{12} = 0, \quad e_{21} = 0, \quad \omega_1 = 0, \quad \omega_2 = 0 \tag{3.4}$$

by virtue of the absolutely rigorous equalities $\varepsilon_i = e_{ii}$ from the relations

$$S^*_{11} = T^*_{11}(1+e_{11}) + T^*_{12}e_{21}, \quad P^*_{11} = Q^*_{11}(1+e_{11}) + Q^*_{12}e_{21}; \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1, 2}}$$

the approximate equalities

$$S^*_{11} \approx T_{11}, \quad P^*_{11} \approx Q_{11}; \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1, 2}}$$

follow with an accuracy of $1+\varepsilon_i \approx 1$, on substituting relations (2.13) and (2.22).

On account of what has been stated above, it is necessary relations (3.3), which hold when $i \neq j$ in the case when $\varepsilon_i \ll 1$, to supplement with the simplified relations

$$\begin{gathered} S^*_{11} \approx S_{11} \approx T_{11} + T_{12}e_{21}, \quad P^*_{11} \approx P_{11} \approx Q_{11} + Q_{12}e_{21}; \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1, 2}} \\ S^*_{i3} \approx S_{i3} = T_{is}\omega_s, \quad P^*_{i3} \approx P_{i3} = Q_{is}\omega_s \end{gathered} \tag{3.5}$$

Relations (3.3) and (3.5) are energetically completely matched with the consistent kinematic relations[3]

$$\begin{gathered} \varepsilon_1 \approx e_1 + (e^2_{12} + \omega^2_1)/2, \quad \varepsilon_2 \approx e_{22} + (e^2_{21} + \omega^2_2)/2 \\ \sin\gamma_{12} \approx 2\varepsilon_{12} = (1+e_{11})e_{21} + (1+e_{22})e_{12} + \omega_1\omega_2 \end{gathered} \tag{3.6}$$

since, when they are used for the variation in the strain potential energy of a shell, we arrive at the expression

$$\begin{gathered} \delta U = \iint_\sigma T^*_{sk}\delta\varepsilon_{sk}d\sigma \approx \iint_\sigma T_{sk}\delta\varepsilon_{sk}d\sigma = \iint_\sigma [T_{11}\delta\varepsilon_1 + T_{22}\delta\varepsilon_2 + \\ + T_{12}\delta(\sin\gamma_{12})] = \iint_\sigma (S_{sk}\delta e_{sk} + S_{s3}\delta\omega_s)d\sigma \end{gathered} \tag{3.7}$$

The forces S_{sk}, S_{i3} are found using formulae (3.3) (when $i \neq j$) and (3.5).

It is necessary to emphasize that, if only "dead" contour forces Q_{ik} act on the shell, then all the above relations and the equilibrium Eq. (2.25) in which $X^*_1 = X^*_3 = 0$ and $S^*_{ij} \approx S_{ij}$, $S^*_{i3} \approx S_{i3}$ hold for arbitrary shear deformations $\sin\gamma_{12}$. When $\mathbf{X}^* \neq 0$, however, the approximate equality $\mathbf{X}^* \approx \mathbf{X}$ only holds under the conditions that

$$1+\varepsilon_i \approx 1, \quad \cos\gamma_{12} \approx 1 \tag{3.8}$$

At the same time, when a "dead" surface load acts on the shell,

$$X_1^* \approx X_1, \quad X_2^* \approx X_2, \quad X_3^* \approx X_3 \tag{3.9}$$

and, in the case of the action of a "following" surface load with specified components Y_1, Y_2, Y_3 in projections onto the directions of the unit vectors $\boldsymbol{l}_i^*$ and $\mathbf{m}^*$ on σ_*,

$$X_1^* \approx X_1 \approx Y_1 + e_{21}Y_2 + E_1Y_3, \quad \overrightarrow{\underleftarrow{1, 2}}, \quad X_3^* \approx X_3 \approx \omega_1 Y_1 + \omega_2 Y_2 + Y_3 \tag{3.10}$$

When equalities (3.4) are satisfied by virtue of the fact that $\varepsilon_i = e_{ii}$, expressions (2.4) and (2.27), which serve to define these vectors, reduce to the equalities

$$\mathbf{l}_i^* = \frac{1+e_{ii}}{1+\varepsilon_i}\mathbf{l}_i \equiv \mathbf{l}_i, \quad \mathbf{m}^* = \frac{E_3}{(1+\varepsilon_1)(1+\varepsilon_2)\cos\gamma_{12}}\mathbf{m} \equiv \mathbf{m} \tag{3.11}$$

Hence, on introducing constraints (3.8), putting

$$\frac{1+e_{ii}}{1+\varepsilon_i} \approx 1, \quad \frac{(1+e_{11})(1+e_{22}) - e_{12}e_{21}}{(1+\varepsilon_1)(1+\varepsilon_2)\cos\gamma_{12}} \approx 1 \tag{3.12}$$

it is necessary to represent expressions (2.4) and (2.27) in the following simplified form

$$\mathbf{l}_1^* \approx \mathbf{l}_1 + e_{12}\mathbf{l}_2 + \omega_1\mathbf{m}, \quad \overrightarrow{\underleftarrow{1, 2}}; \quad \mathbf{m}^* \approx E_i\mathbf{l}_i + \mathbf{m} \tag{3.13}$$

which enables as, when equalities (3.4) are satisfied, to take the limit to equalities (3.11) which leads, in accordance with the representations $\boldsymbol{X} = Y_i\boldsymbol{l}_i^* + Y_3\boldsymbol{m}^*$, to relations (3.10).

By analogy with expressions (3.13), when constraints (3.8) are introduced, it is also necessary to simplify expressions (2.34). In the special case, when equalities (3.4) are satisfied, the equalities

$$\mathbf{n}_1^* = \frac{E_3(1+e_{22})}{\Delta_{12}(1+\varepsilon_2)}\mathbf{l}_1 \equiv \mathbf{l}_1, \quad \overrightarrow{\underleftarrow{1, 2}} \tag{3.14}$$

follow from expressions (2.34) when relations (2.35) and (2.28) are taken into account.

In the case of constraints (3.8), the approximate relations

$$\mathbf{n}_1^* \approx \mathbf{l}_1 + n_{12}\mathbf{l}_2 + n_{12}\mathbf{m}, \quad \overrightarrow{\underleftarrow{1, 2}} \tag{3.15}$$

which follow from relations (2.34) when the following constraints are introduced

$$\frac{n_{11}}{\Delta_{12}(1+\varepsilon_2)} \approx 1, \quad \overrightarrow{\underleftarrow{1, 2}} \tag{3.16}$$

also reduce to these equalities.

Now, using approximate equalities (3.13) and (3.15) for $\boldsymbol{P}_i^* \approx \boldsymbol{P}_i$, we obtain the representations

$$\mathbf{P}_i = P_{ik}\mathbf{l}_k + P_{i3}\mathbf{m},$$

where

$$P_{11} = Q_1 + Q_{12}e_{21}, \quad P_{12} = Q_1n_{12} + (1+e_{22})Q_{12}, \quad P_{13} = Q_1n_{13} + Q_{12}\omega_2; \quad \overrightarrow{\underleftarrow{1, 2}} \tag{3.17}$$

It earlier seen that the above relations also follow from relations (2.36) when constraints (3.16) and (3.8) are introduced.

In concluding this section for the case of small deformations, we consider the variational equation of the principle of possible displacements, constructed earlier (Ref.,[3] formula (1.18)) in which the components of the surface and contour loads are determined using the formulae presented above, depending on their types. After standard reductions, this equation takes the form

$$\int\limits_{x_-^{3-s}}^{x_+^{3-s}} [(S_{sk} - P_{sk})\delta u_k + (S_{s3} - P_{s3})\delta w]A_{3-s}dx^{3-s}\Big|_{x^s = x_-^s}^{x^s = x_+^s} -$$

$$-\iint\limits_{\sigma}(f_s\delta u_s + f_3\delta w)dx^1dx^2 = 0 \tag{3.18}$$

whence follow the previously constituted[3] equilibrium equations and static boundary conditions on the contour lines $x^i = x_-^i$, $x^i = x_+^i$, which are obtained from relations (2.25) by discarding the asterisks on all of the internal and external forces.

For small deformations of a shell made of an elastic orthotropic material, it is permissible to relate the stresses σ_{11}, σ_{22}, σ_{12} to the deformations ε_1, ε_2, $\sin\gamma_{12}$ by means of the standard generalized Hooke's law relations

$$\sigma_{11} = \tilde{E}_1(\varepsilon_1 + \upsilon_{21}\varepsilon_2), \quad \underleftarrow{\overrightarrow{1, 2}}; \quad \sigma_{12} = 2G_{12}\sin\gamma_{12} \tag{3.19}$$

if the directions of orthotropy coincide with the lines of principal curvatures in the middle surface σ. When they are used for the forces T_{ij}, we arrive at the elasticity relations

$$T_{11} = B_{11}(\varepsilon_1 + \upsilon_{21}\varepsilon_2), \quad \underleftarrow{\overrightarrow{1, 2}}; \quad T_{12} = 2B_{12}\sin\gamma_{12} \tag{3.20}$$

where $B_{ii} = t\tilde{E}_i$ are the extension-compression stiffnesses and $B_{12} = tG_{12}$ is the shear stiffness.

Note that relations (3.19) and (3.20), which have been formulated for a linear elastic material, also hold in the case of finite deformations if the thickness t is replaced by $t^* = t(1 + \varepsilon_3)$. However, the answer to the question whether it is possible to use these relations to solve specific problems cannot be unambiguous in all cases since, instead of (3.19), Hooke's law can also be formulated for the generalized components of the stresses in the form

$$\sigma_{11}^* = \tilde{E}_1(\varepsilon_1 + \upsilon_{21}\varepsilon_2), \quad \underleftarrow{\overrightarrow{1, 2}}; \quad \sigma_{12}^* = \sigma_{12} = 2G_{12}\sin\gamma_{12} \tag{3.21}$$

if they correspond more to the experimental results. On account of what has been said, relations (3.19) and (3.20), as is assumed for the absolute majority of materials, can only be used within the limits of small elastic deformations when the equalities $\sigma_{ik}^* \approx \sigma_{ik}$ are approximately satisfied.

4. The equations of the neutral equilibrium and perturbed motion of momentless shells in the case of small deformations

We will now consider two equilibrium states of a shell. Suppose the first of them, which is unperturbed, is characterized by internal forces T_{11}^0, T_{22}^0, $T_{12}^0 = T_{21}^0$, external surface forces X_1^0, X_2^0, X_3^0 which are applied to the middle surface σ, and linear forces P_{ik}^0, P_{i3}^0, applied to the contour lines $x^i = x_-^i$, $x^i = x_+^i$. Making the standard assumption that the shell is stressed but not deformed in the first state, by linearising the non-linear equations formulated in the preceding section in the neighbourhood of the unperturbed state and retaining the earlier notation for the increments in the parameters of the stress-strain state and the external forces of the perturbed state, we arrive at the following system of linearized neutral equilibrium equations

$$\begin{aligned} f_1 &= \frac{\partial(A_1S_{11})}{\partial x^1} + \frac{\partial(A_1S_{21})}{\partial x^2} - \frac{\partial A_2}{\partial x^1}S_{22} + \frac{\partial A_1}{\partial x^2}S_{12} + A_1A_2(k_1S_{13} - X_1) = 0, \quad \underleftarrow{\overrightarrow{1, 2}} \\ f_3 &= \frac{\partial(A_2S_{13})}{\partial x^1} + \frac{\partial(A_1S_{23})}{\partial x^2} - A_1A_2(k_1S_{11} + k_2S_{22} - X_3) = 0 \end{aligned} \tag{4.1}$$

where

$$S_{11} = T_{11} + T_{12}^0e_{21}, \quad S_{12} = T_{11}^0e_{12} + T_{12}^0e_{22} + T_{12}, \quad S_{i3} = T_{11}^0\omega_1 + T_{12}^0\omega_2; \quad \underleftarrow{\overrightarrow{1, 2}} \tag{4.2}$$

and, in unlike relations (3.22),

$$T_{11} = B_{11}(e_{11} + \upsilon_{21}e_{22}), \quad \underleftarrow{\overrightarrow{1, 2}}; \quad T_{12} = B_{12}(e_{12} + e_{21}) \tag{4.3}$$

Eq. (4.1) have been formulated earlier[3] for $X_i = 0$, $X_3 = 0$ and correspond to the action of surface forces of constant direction. In them, it is necessary to take

$$X_1 = e_{21}Y_2^0 - \omega_1Y_3^0, \quad X_2 = e_{12}Y_1^0 - \omega_2Y_3^0, \quad X_3 = \omega_1Y_1^0 + \omega_2Y_2^0 \tag{4.4}$$

in the case of the action of "following" surface loads Y_i^0 and Y_3^0.

In the case of Eq. (4.1), which have been set up on the contour lines $x^i = x_-^i$, $x^i = x_+^i$, boundary conditions of the form

$$S_{i1} = P_{ij} \text{ when } \delta u_j \neq 0, \quad S_{i3} = P_{i3} \text{ when } \delta w \neq 0 \tag{4.5}$$

are formulated in which $P_{i\alpha} = 0$ in the case of the action of "dead" contour forces Q_{ik}^0. Under the action of "following" contour forces of the first type Q_{11}^0, Q_{22}^0, Q_{12}^0, Q_{21}^0

$$P_{11} = Q_{12}^0e_{21}, \quad P_{12} = Q_{11}^0e_{12} + Q_{12}^0e_{22}, \quad P_{13} = Q_{11}^0\omega_1 + Q_{12}^0\omega_2, \quad \underleftarrow{\overrightarrow{1, 2}} \tag{4.6}$$

and, under the action of following forces of the second type

$$P_{11} = Q_{12}^0e_{21}, \quad P_{12} = -Q_1^0e_{21} + Q_{12}^0e_{22}, \quad P_{13} = Q_1^0\omega_1 + Q_{12}^0\omega_2, \quad \underleftarrow{\overrightarrow{1, 2}}$$

When non-conservative forces act on the shell, instead of the equation

$$\iint_{\sigma}(S_{sk}\delta e_{sk}+S_{s3}\delta\omega_s)d\sigma = 0 \tag{4.7}$$

(the forces S_{sk} and S_{s3} are defined by formulae (4.2)) it is necessary to use the variational equation (ρ is the density of the shell material)

$$\int_{t_0}^{t_1}\sum_{s=1}^{2}\left\{\iint_{\sigma}\left[S_{sk}\delta e_{sk}+S_{s3}\delta\omega_s-\left(X_s-t\rho\frac{\partial^2 u_s}{\partial t^2}\right)\delta u_s-\left(X_3-t\rho\frac{\partial^2 w}{\partial t^2}\right)\delta w\right]d\sigma - \right.$$
$$\left. - \int_{x_-^{3-s}}^{x_+^{3-s}}(P_{sk}\delta u_k+P_{s3}\delta w)A_{3-s}dx^{3-s}\Bigg|_{x^s=x_-^s}^{x^s=x_+^s}\right\}dt = 0 \tag{4.8}$$

from which, instead of the neutral equilibrium equations (4.1), we obtain equations of the perturbed motion of the form

$$f_i - t\rho A_1A_2\frac{\partial^2 u_i}{\partial t^2} = 0,\quad f_3 - t\rho A_1A_2\frac{\partial^2 w}{\partial t^2} = 0 \tag{4.9}$$

5. Conclusion

The resolvents of the geometrically non-linear theory of elasticity and momentless shells, formulated for the case of small deformations using the proposed consistent relations, differ from the analogous equations formulated using the classical relations given in the literature when a number of non-linear terms, which are extremely small compared with the other terms, are not present. When they are discarded in the equations of the classical non-linear theory of elasticity, very small perturbations are introduced which, as a number of investigations carried out on the numerical solution of the number of problems on the geometrically non-linear deformation and stability of straight beams have shown, under certain forms of loading lead to considerable perturbations (more than 20%) in the solutions determining both the parameters of the subcritical stress-strain state as well as the values of the critical loads in the direction of understating them. The results of such investigations are in complete agreement with the determination of the stability of mechanical systems. They have been partially reflected earlier.[11]

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